Old Higher Qu with AH content

1 In certain topics in Mathematics, such as calculus, we often require to write an expression such as $\dfrac{8x+1}{(2x+1)(x-1)}$ in the form $\dfrac{2}{2x+1}+\dfrac{3}{x-1}$.

$\dfrac{2}{2x+1}+\dfrac{3}{x-1}$ are called **Partial Fractions** for $\dfrac{8x+1}{(2x+1)(x-1)}$.

The worked example shows you how to find partial fractions for the expression $\dfrac{6x+2}{(x+2)(x-3)}$.

Worked Example

Find partial fractions for $\dfrac{6x+2}{(x+2)(x-3)}$.

Let $\dfrac{6x+2}{(x+2)(x-3)}=\dfrac{A}{x+2}+\dfrac{B}{x-3}$ where A and B are constants

$$=\frac{A(x-3)}{(x+2)(x-3)}+\frac{B(x+2)}{(x-3)(x+2)}$$

i.e. $\dfrac{6x+2}{(x+2)(x-3)}=\dfrac{A(x-3)+B(x+2)}{(x+2)(x-3)}$

Hence $6x+2=A(x-3)+B(x+2)$ for all values of x.

A and B can be found as follows:

Select a value of x that makes the first bracket zero

Let $x=3$ (this eliminates A)

$18+2=A\times 0 \; + \; B\times 5$

$20=5B$

$\underline{B=4}$

Select a value of x that makes the second bracket zero

Let $x=-2$ (this eliminates B)

$-12+2=A\times(-5) \; + \; B\times 0$

$-10=-5A$

$\underline{A=2}$

Therefore $\dfrac{6x+2}{(x+2)(x-3)}=\dfrac{2}{x+2}+\dfrac{4}{x-3}$.

Find partial fractions for $\dfrac{5x+1}{(x-4)(x+3)}$. **(6)**

2

A system of 3 equations in 3 unknowns can be solved by a method known as Gaussian Elimination as shown below.

Example

Solve the system of equations by Gaussian Elimination.

$$\begin{aligned} x + 2y - 3z &= 11 \\ 2x + 2y - z &= 11 \\ 3x - 2y + 4z &= -4 \end{aligned}$$

A Write out the coefficients in an array:

- Row 1
- Row 2
- Row 3

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 2 & 2 & -1 & 11 \\ 3 & -2 & 4 & -4 \end{array}\right)$$

B Keep Row 1 the same. Make Row 2 and Row 3 each begin with a zero by subtracting multiples of Row 1 from them.

- Row 1 is kept the same
- Row 2 becomes 'Row 2 – 2 x Row 1'
- Row 3 becomes 'Row 3 – 3 x Row 1'

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ \mathbf{0} & \mathbf{-2} & \mathbf{5} & \mathbf{-11} \\ \mathbf{0} & \mathbf{-8} & \mathbf{13} & \mathbf{-37} \end{array}\right)$$

C Keep Row 1 and Row 2 the same. Make Row 3 begin with two zeros, by subtracting a multiple of Row 2 from it.

- Row 1 is kept the same
- Row 2 is kept the same
- Row 3 becomes 'Row 3 – 4 x Row 2'

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 11 \\ 0 & -2 & 5 & -11 \\ \mathbf{0} & \mathbf{0} & \mathbf{-7} & \mathbf{7} \end{array}\right) \begin{array}{l} \text{.........(1)} \\ \text{.........(2)} \\ \text{.........(3)} \end{array}$$

D

- Line (3) gives $-7z = 7, \quad z = -1$
- Line (2) gives $-2y + 5z = -11$
  $-2y + (-5) = -11, \quad y = 3$
- Line (1) gives $x + 2y - 3z = 11$
  $x + 6 + 3 = 11, \quad x = 2$

So the solution is $x = 2, y = 3, z = -1$

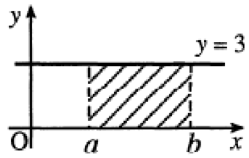
Solve the following system of equations by Gaussian Elimination **as shown above.**

$$\begin{aligned} x - 2y + z &= 6 \\ 3x + y - z &= 7 \\ 4x - y + 2z &= 15 \end{aligned}$$

(7)

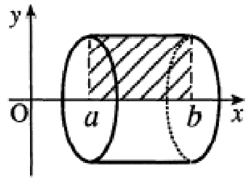
3

Diagram 1 shows the area between the line $y = 3$ and the x-axis from $x = a$ to $x = b$. If this area is rotated through 360° about the x-axis, it forms a solid shape (a cylinder) as shown in Diagram 2.



Diag.1

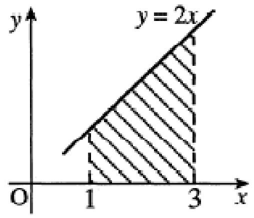
The volume of this solid may be obtained by evaluating the integral $\pi \int_a^b y^2 \, dx$.



Diag.2

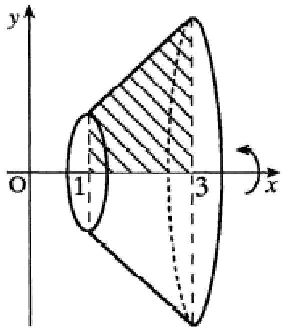
Worked Example

The area between $y = 2x$ and the x-axis from $x = 1$ to $x = 3$ is rotated about the x-axis. The volume of the solid is calculated as follows:

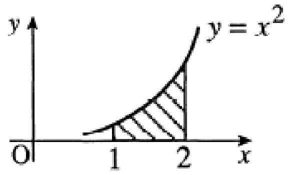


$$y = 2x$$
$$y^2 = (2x)^2 = 4x^2$$

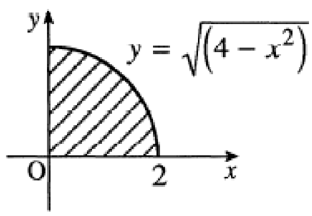
$$\pi \int_1^3 y^2 \, dx$$
$$= \pi \int_1^3 4x^2 \, dx$$
$$= \pi \left[\tfrac{4}{3} x^3\right]_1^3$$
$$= \pi \left[36 - \tfrac{4}{3}\right]$$
$$\text{Volume} = \tfrac{104}{3}\pi \text{ units}^3$$



(*a*) Use this method to find the volume of the solid formed when the area between $y = x^2$ and the x-axis from $x = 1$ to $x = 2$ is rotated about the x-axis. **(4)**



(*b*) (i) Use this method to find the volume of the solid formed when the area between $y = \sqrt{(4 - x^2)}$ and the x-axis from $x = 0$ to $x = 2$ is rotated about the x-axis. **(4)**



(ii) **Hence** write down the volume of a sphere of radius 2. **(1)**

4

EXAMPLE

(i) Let $f(x) = x^3 + 5x - 1$.
Since $f(0) = -1$ and $f(1) = 5$
the equation $f(x) = 0$ has a root in the interval $0 < x < 1$ because $f(0) < 0$ and $f(1) > 0$.

(ii) To find this root, the equation $x^3 + 5x - 1 = 0$ can be rearranged as follows :

$$x^3 + 5x - 1 = 0$$
$$x^3 + 5x = 1$$
$$x(x^2 + 5) = 1$$
$$x = \frac{1}{x^2 + 5}$$

We can write this result as a recurrence relation

$$x_{n+1} = \frac{1}{x_n^2 + 5}$$

and use it to find this root. In this example we will work to 3 decimal places and can therefore give the final answer to 2 decimal places.

(iii) For our first estimate, x_1, we use the mid-point of the interval $0 < x < 1$ [from part (i)].

$$x_1 = 0\cdot 5, \qquad x_2 = \frac{1}{0\cdot 5^2 + 5} = 0\cdot 190$$
$$x_2 = 0\cdot 190 \qquad x_3 = \frac{1}{0\cdot 190^2 + 5} = 0\cdot 199$$
$$x_3 = 0\cdot 199 \qquad x_4 = \frac{1}{0\cdot 199^2 + 5} = 0\cdot 198$$
$$x_4 = 0\cdot 198 \qquad x_5 = \frac{1}{0\cdot 198^2 + 5} = 0\cdot 198$$

Hence, correct to 2 decimal places, the root is $x = 0{\cdot}20$.

(*a*) Show that the equation $2x^3 + 3x - 1 = 0$ has a root in the interval $0 < x < 0{\cdot}5$. **(2)**

(*b*) By using the technique described above find this root correct to 2 decimal places. **(6)**

5

An array of numbers such as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a matrix. The eigenvalues of the matrix

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are defined to be the roots of the equation $(a-x)(d-x)-bc=0$.

Example

> In order to find the eigenvalues of the matrix $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$
>
> solve $\quad (1-x)(2-x)-4x3=0$
>
> solution:
> $$2-3x+x^2-12=0$$
> $$x^2-3x-10=0$$
> $$(x+2)(x-5)=0$$
> $$x=-2 \text{ or } x=5$$
>
> so the eigenvalues of **B** are −2 and 5

(*a*) Find the eigenvalues of $\mathbf{C} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$. **(3)**

(*b*) Find the value of *t* for which the eigenvalues of the matrix $\mathbf{D} = \begin{pmatrix} 3 & -1 \\ t & 1 \end{pmatrix}$ are equal. **(5)**

6

A function *f* can be expressed as an infinite series by $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\ldots\ldots$

(*a*) Write down the series for $f(2x)$ as far as the term in x^5. 1

The derivative of $f(x)$ can be calculated as follows:

> $$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\ldots\ldots$$
> so $$f'(x) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \frac{5x^4}{120} + \ldots\ldots\ldots$$
> $$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots\ldots$$
> *i.e.* $f'(x) = f(x)$

(*b*) If $g(x) = f(2x)$ find $g'(x)$ and express it in terms of $f(2x)$. 3

7

There is a rule known as the Product Rule which is used, as shown below, to differentiate any product of two functions of the same variable.

> **The Product Rule**
>
> If $P(x) = f(x).g(x)$, then $P'(x) = f'(x).g(x) + f(x).g'(x)$
>
> **Example:** Find the derivative of $P(x) = x^2 \sin x$.
>
> $P(x) = x^2 \sin x \qquad$ Choose $\quad f(x) = x^2$ *and* $g(x) = \sin^2 x$
>
> then $\quad f'(x) = 2x$ *and* $g'(x) = \cos x$
>
> so $\quad P'(x) = 2x.\sin x + x^2.\cos x$
>
> $P'(x) = 2x \sin x + x^2 \cos x$

Use the Product Rule to find the derivative of $P(x) = x^3 \cos x$ **(5)**

8

Polynomial equations often have roots which are not whole numbers.
One method of estimating the roots of such equations is to make repeated use of the following:

> If $x = p$ is an estimate of a root of the equation $f(x) = 0$, then $x = q$ will will be a closer estimate where $q = p - \dfrac{f(p)}{f'(p)}$.

> **Example**
>
> One of the roots of the equation $x^2 - 2x - 5 = 0$ is known to lie between 3 and 4.
>
> We have $f(x) = x^2 - 2x - 5$ and so $f'(x) = 2x - 2$.
>
> **Choose p = 3** (1st estimate) then $q = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{-2}{4} = 3.5$.
>
> **Choose p = 3.5** (2nd estimate) then $q = 3.5 - \frac{f(3.5)}{f'(3.5)} = 3.5 - \frac{0.25}{5} = 3.45$.
>
> **Choose p = 3.45** (3rd estimate) then $q = 3.45 - \frac{f(3.45)}{f'(3.45)} = 3.45 - \frac{0.0025}{4.9} = 3.449$.
>
> **Conclusion** The root, correct to 1 decimal place, is **x = 3.4**

(*a*) Show that the equation $x^3 - 2x^2 + 6x - 4 = 0$ has a root between 0 and 1. **(3)**

(*b*) Use the method described above to find this root correct to 1 decimal place. **(6)**

9

A function f is **EVEN** if $f(-x) = f(x)$
e.g. when $f(x) = x^2$, f is **EVEN** because $f(-x) = (-x)^2 = x^2 = f(x)$.

A function f is **ODD** if $f(-x) = -f(x)$
e.g. when $f(x) = x^3$, f is **ODD** because $f(-x) = (-x)^3 = -x^3 = -f(x)$.

(*a*) Given that $g(x) = \cos x$ and $h(x) = \sin 2x$, decide for each of the functions g and h whether it is **EVEN** or **ODD**.
Justify your decisions. **(4)**

(*b*) Evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; dx$ and $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x \; dx$. **(5)**

(*c*) On separate diagrams, draw rough sketches of the graphs of $y = \cos x$ and $y = \sin 2x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **(2)**

(*d*) If $v(x) = x\cos x$, check whether the function v is **EVEN** or **ODD** and suggest a value for $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x \; dx$. **(2)**

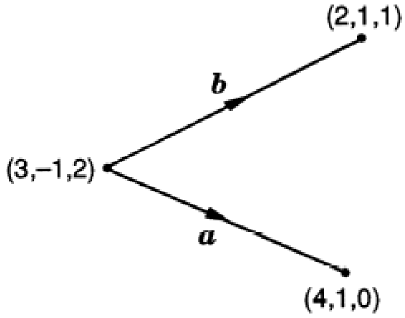
10

The *vector product*, $\boldsymbol{a} \times \boldsymbol{b}$, of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \text{ where } \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Example

when $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ then $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 2\times2-3\times0 \\ 3\times(-1)-1\times2 \\ 1\times0-2\times(-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$



(*a*) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are as shown in the diagram and $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$, evaluate $\boldsymbol{c}$. **(3)**

(*b*) By considering $\boldsymbol{a.c}$ and $\boldsymbol{b.c}$, what can be concluded about $\boldsymbol{c}$? **(4)**

1

- $\bullet^1$ $\frac{A}{x-4} + \frac{B}{x+3}$
- $\bullet^2$ $\frac{A(x+3)+B(x-4)}{(2x-1)(x+3)}$
- $\bullet^3$ $5x + 1 = A(x+3) + B(x-4)$
- $\bullet^4$ choose to let $x = -3$ and 4 in turn
- $\bullet^5$ $A = 3$
- $\bullet^6$ $B = 2$

2	$\bullet^1$ $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 6 \\ 3 & 1 & -1 & 7 \\ 4 & -1 & 2 & 15 \end{array}\right)$ $\bullet^2$ $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 6 \\ 0 & 7 & -4 & -11 \end{array}\right)$ $\bullet^3$ $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 6 \\ 0 & 7 & -4 & -11 \\ 0 & 7 & -2 & -9 \end{array}\right)$ $\bullet^4$ $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 6 \\ 0 & 7 & -4 & -11 \\ 0 & 0 & 2 & 2 \end{array}\right)$ $\bullet^5$ $2z = 2,\ z = 1$ $\bullet^6$ $7y - 4z = -11,\ y = -1$ $\bullet^7$ $x - 2y + z = 6,\ \ x = 3$
3	(*a*) $\bullet^1$ $y^2 = x^4$ $\bullet^2$ $\pi\int_1^2 x^4\ dx$ $\bullet^3$ $\pi\left[\frac{1}{5}x^5\right]_1^2$ $\bullet^4$ $\frac{31}{5}\pi$ (*accept* 19.5) (*b*) $\bullet^5$ $y^2 = 4 - x^2$ $\bullet^6$ $\pi\int_0^2\left(4 - x^2\right)\ dx$ $\bullet^7$ $\pi\left[4x - \frac{1}{3}x^3\right]_0^2$ $\bullet^8$ $\frac{16}{3}\pi$ (*c*) $\bullet^9$ $\frac{32}{3}\pi$ **or** $2\times\frac{16}{3}\pi$
4	(*a*) $\bullet^1$ $f(0) = -1$ **and** $f(0\cdot5) = 0\cdot75$ $\bullet^2$ "$f(0) < 0$ and $f(0.5) > 0$" or equiv. **explicitly stated** (*b*) $\bullet^3$ $x = \frac{1}{2x^2+3}$ $\bullet^4$ $x_1 = 0.25$ $\bullet^5$ $x_2 = 0.32$ $\bullet^6$ $x_3 = 0.312$ rounded to 3dp $\bullet^7$ $x_4 = 0.313$ *and* $x_5 = 0.313$ $\bullet^8$ 0.31 correct to 2dp
5	(*a*) $\bullet^1$ $(3-x)(5-x) - 2\times4 = 0$ $\bullet^2$ $x^2 - 8x + 7 = 0$ $\bullet^3$ eigenvalues are $1,\ 7$ (*b*) $\bullet^4$ $(3-x)(1-x) + t = 0$ $\bullet^5$ $x^2 - 4x + (3+t) = 0$ $\bullet^6$ $\Delta = 0$ for equal roots or equiv. $\bullet^7$ $\Delta = 16 - 4\times1\times(3+t)$ or equiv. $\bullet^8$ $t = 1$
6	$\bullet^1$ $1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \frac{(2x)^4}{24} + \frac{(2x)^5}{120}$ $\bullet^2$ $1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5$ $\bullet^3$ $2 + 4x + 4x^2 + \frac{8}{3}x^3 + \frac{4}{3}x^4$ $\bullet^4$ $2\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4\right)$ ***and*** $2f(2x)$

7	•1 $f(x)=x^3$ •2 $g(x)=\cos x$ •3 $f'(x)=3x^2$ **and** $g(x)=-\sin x$ •4 $3x^2\cos x$ •5 $-x^3\sin x$
8	(*a*) •1 $0^3-2\times0+6\times0-4=-4$ •2 $1^3-2\times1+6\times1-4=1$ •3 $f(0)<0$ and $f(1)>0$ so $0<\text{root}<1$ (*b*) •4 $f'(x)=3x^2-4x+6$ •5 *e.g.* 1st est $=0$, 2nd est $=0-\frac{f(0)}{f'(0)}=0.67$ •6 3rd est $=0.67-\frac{f(0.67)}{f'(0.67)}$ •7 0.7936 •8 4th est $=0.7936-\frac{f(0.7936)}{f'(0.7936)}=0.7932$ •9 0.8
9	(*a*) •1 $\cos(-x)=\cos x$ •2 g is EVEN •3 $\sin(-2x)=-\sin(2x)$ •4 h is ODD (*b*) •5 $\sin x$ •6 $[\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2$ •7 $-\cos 2x$ •8 $\times\frac{1}{2}$ •9 $\left[-\frac{1}{2}\cos 2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=0$ (*c*) •10 sketch of $g(x)=\cos x$ •11 sketch of $h(x)=\sin 2x$ (*d*) •12 $v(x)$ is ODD •13 0
10	**(*a*)** •1 $a=\begin{pmatrix}1\\2\\-2\end{pmatrix}$ **and** $b=\begin{pmatrix}-1\\2\\-1\end{pmatrix}$ •2 substitute in the rule for **a** x **b** •3 answer **(*b*)** •4 evaluate ***a.c*** •5 evaluate ***b.c*** •6 a statement that ***a*** is perpendicular to ***c*** •7 a statement that ***b*** is perpendicular to ***c***